Measurement Systems

Measurement Errors

Applied Mechatronics

Module 2.2

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Systematic Errors (1)

- Definition
  - Static error due to errors in the measurement setup
- Characteristics
  - Have always the same magnitude
  - Can be determined
  - Can be compensated

Systematic Errors (2)

- Origin
  - System Disturbance
    - The measurement has an impact to the system
  - Environmental Inputs
    - Ambient temperature
    - Electromagnetic pollution
    - Position or orientation of the instrument
  - Wiring
    - Hidden resistances

Systematic Errors (3)

- Origin (continued)
  - Aging and Wear
    - System decay
- Reduction
  - Careful measurement setup
  - Differential inputs
    - Compensates environmental inputs
  - Calibration

Systematic Errors (4)

- Error Propagation
  - Quantity depends on several measurements
    $$y = \mathrm{f}(x_1, x_2, \dots)$$
  - Each measurement has an uncertainty
    $$x_i \pm \Delta x_i$$
    Worst Case
  - Uncertainty of quantity is
    $$\Delta \mathrm{f} = \left|\frac{\partial f}{\partial x_1} \cdot \Delta x_1\right| + \left|\frac{\partial f}{\partial x_2} \cdot \Delta x_2\right| + \cdots$$

Systematic Errors (5)

- Error Propagation Example
  - Resistance depends on voltage and current
    - $R = \frac{V}{I}$
  - Voltage and current have uncertainty
    - $V = 0.9 \ldots 1.1\,V \rightarrow V = 1.0 \pm 0.1\,V$
    - $I = 0.95 \ldots 1.05 \mu A \rightarrow I = 1.0 \pm 0.05\,\mu A$
  - Derivatives
    - $\frac{\partial R}{\partial V} = \frac{1}{I}$

Maybe caused by tolerances

Systematic Errors (6)

- Error Propagation Example (continued)
  - Derivatives (continued)
    - $\frac{\partial R}{\partial I} = -\frac{V}{I^2}$
  - Worst systematic error
    - $\Delta \mathrm{R} = \left|\frac{\partial R}{\partial V} \cdot \Delta V\right| + \left|\frac{\partial R}{\partial I} \cdot \Delta I\right|$
    - $\Delta \mathrm{R} = \left|\frac{1}{I} \cdot \Delta V\right| + \left|-\frac{V}{I^2} \cdot \Delta I\right|$
    - $\Delta \mathrm{R} = \left|\frac{1}{1\ \mu A} \cdot 0.1\ V\right| + \left|-\frac{1.0\ V}{(1\ \mu A)^2} \cdot 0.05\ \mu A\right|$

Systematic Errors (5)

- Error Propagation Example (finished)
  - Worst systematic error (continued)
    - $\Delta\mathrm{R} = 10^5\ \Omega + 5 \cdot 10^4\ \Omega$
    - $\Delta\mathrm{R} = 150\ k\Omega$
  - Measured value
    - $\mathrm{R} = \frac{V}{I} = \frac{1\ V}{1\ \mu A} = 1\ M\Omega$
  - Measured value with uncertainty
    - $\mathrm{R} = 1\ M\Omega \pm 150\ k\Omega$

Random Errors (1)

- Definition
  - Unpredictable variations in measurement due to small perturbations
- Characteristics
  - Magnitude and direction of error varies
    - Positive as well as negative deviations
  - Cannot be determined
  - Can be compensated by averaging

Random Errors (2)

- Averaging
  - Mean Value
    - $\bar{x} = \frac{x_1 + x_2 + \cdots + x_n}{n}$
    - Averaging compensates random variations
    - Mean value is the better the closer the values
  - Median
    - $\bar{x}_{med} = \begin{cases} x_{\left(\frac{n+1}{2}\right)}, \text{if } n \text{ is odd} \\ \frac{x_{\left(\frac{n}{2}\right)} + x_{\left(\frac{n}{2}+1\right)}}{2}, \text{if } n \text{ is even} \end{cases}$

Value in the sorted middle

Mean value of the values in the sorted middle

Random Errors (3)

- Averaging (continued)
  - Standard Deviation
    - $d_i = x_i - \bar{x}$ (deviation)
    - $\sigma = \sqrt{\frac{d_1^2 + d_2^2 + \cdots + d_n^2}{n-1}}$ (standard deviation)
    - Confidence interval
      - $[\bar{x} - \sigma, \bar{x} + \sigma]$: comprises 68% of all values
      - $[\bar{x} - 2\sigma, \bar{x} + 2\sigma]$: comprises 95.4% of all values
      - $[\bar{x} - 3\sigma, \bar{x} + 3\sigma]$: comprises 99.7% of all values

Random Errors (4)

- Error Propagation
  - Quantity depends on several measurements

$$y = \mathrm{f}(\bar{x}_1, \bar{x}_2, \dots)$$

  - Each measurement has an uncertainty

$$x_i \pm \Delta x_i$$

  - Uncertainty of quantity is

$$\Delta\mathrm{f} = \sqrt{\left|\frac{\partial f}{\partial x_1} \cdot \Delta\bar{x}_1\right|^2 + \left|\frac{\partial f}{\partial x_2} \cdot \Delta\bar{x}_2\right|^2 + \cdots}$$

Random Errors (5)

- Error Propagation Example
  - Resistance depends on voltage and current
    - $R = \frac{V}{I}$
  - Voltage and current have uncertainty
    - $V = 0.9 \dots 1.1\,V \rightarrow V = 1.0 \pm 0.1\,V$
    - $I = 0.95 \dots 1.05 \mu A \rightarrow I = 1.0 \pm 0.05\,\mu A$
  - Derivatives
    - $\frac{\partial R}{\partial V} = \frac{1}{I}$

Random Errors (6)

- Error Propagation Example (continued)
  - Derivatives (continued)
    - $\frac{\partial R}{\partial I} = -\frac{V}{I^2}$
  - Random error of the measured quantitiy
    - $\Delta \mathrm{R} = \sqrt{\left|\frac{\partial R}{\partial V} \cdot \Delta V\right|^2 + \left|\frac{\partial R}{\partial I} \cdot \Delta I\right|^2}$
    - $\Delta \mathrm{R} = \sqrt{\left|\frac{1}{I} \cdot \Delta V\right|^2 + \left|-\frac{V}{I^2} \cdot \Delta I\right|^2}$
    - $\Delta \mathrm{R} = \sqrt{\left|\frac{1}{1\ \mu A} \cdot 0.1\ V\right|^2 + \left|-\frac{1.0\ V}{(1\ \mu A)^2} \cdot 0.05\ \mu A\right|^2}$

Random Errors (7)

- Error Propagation Example (finished)
  - Worst systematic error (continued)
    - $\Delta R = \sqrt{(10^5\ \Omega)^2 + (5 \cdot 10^4\ \Omega)^2}$
    - $\Delta R = 112\ k\Omega$
  - Measured value
    - $R = \frac{V}{I} = \frac{1\ V}{1\ \mu A} = 1\ M\Omega$
  - Measured value with statistical uncertainty
    - $R = 1\ M\Omega \pm 112\ k\Omega$

Appendix (1)

- Notation of Derivatives
  - Lagrange's notation
    - $f'(x)$
  - Leibniz's notation
    - $\frac{df(x)}{dx}$
  - Partial derivative
    - $\frac{\partial f(x,y,\cdots)}{\partial x}, \frac{\partial f(x,y,\cdots)}{\partial y}, \ldots$

Appendix (2)

- Derivative of some Functions
  - $(a)' = 0$
  - $(x)' = 1$
  - $(x^2)' = 2x$
  - $\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$
  - $(\sqrt{x})' = \frac{1}{2\sqrt{x}}$
  - $(x^n)' = n \cdot x^{n-1}$

Appendix (3)

- Differentiation Rules
  - Constant factor rule
    - $(\mathrm{a} \cdot f)' = \mathrm{a} \cdot f'$
  - Sum rule
    - $(f + g)' = f' + g'$
  - Product rule
    - $(f \cdot g)' = f' \cdot g + f \cdot g'$